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On Mullineux Symbols

CHRISTINE BESSENRODT

Fakultät für Mathematik, Otto-von-Guericke-Universität, 39016 Magdeburg, Germany

AND

JØRN B. OLSSON

Matematisk Institut, Københavns Universitet, Universitetsparken 5, 2100 København Ø, Denmark

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The Mullineux symbols of a special class of p-regular partitions are classified. As a consequence it is shown that two apparently very difficult conjectures in the modular representation theory of finite symmetric groups are compatible. © 1994 Academic Press, Inc.

In 1979 Mullineux defined an interesting involutory bijection on the set of p-regular partitions of a positive integer n, which may be seen as the p-analogue of the conjugation map on general partitions of n. Apart from the properties proved by Mullineux [6, 7] very little is known about this map, which may be described by associating to a p-regular partition a Mullineux symbol and then defining a map on these symbols. In this paper we classify the Mullineux symbols of a special class of p-regular partitions. As a consequence we prove the compatibility of two conjectures in the modular representation theory of the finite symmetric groups S_n. These two conjectures are difficult counterparts to well-known results about ordinary irreducible characters in S_n for modular characters.

We have the following results for ordinary characters in mind: the irreducible characters χ_λ of S_n are labelled by partitions $\lambda = (l_1, l_2, \ldots, l_m)$ $(l_1 + \cdots + l_m = n,\ l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ of n.

(I) Multiplying the character χ_λ of S_n by the sign character gives the character χ_{λ°, where λ° is the partition conjugate to λ [3, 2.1.8].

(II) The character χ_λ remains irreducible when restricted to S_{n-1} if and only if all parts l_i of λ are equal [3, 2.4.3].

Let p be a prime number and consider the modular representations of S_n in characteristic p.

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The modular irreducible characters ρ_λ of S_n may be labelled by *p-regular* partitions λ of n, a partition being p-regular if no part is repeated p (or more) times [3, 6.1]; this is the labelling we will consider in the sequel.

Multiplying the modular character ρ_λ of S_n by the sign character of S_n gives another modular irreducible character, labelled by a p-regular partition λ^P. We are then faced with the following problems.

Problem I. Describe the involutory map $\lambda \mapsto \lambda^P$ on the set of p-regular partitions.

Problem II. Determine those p-regular partitions λ where the restriction of ρ_λ to S_{n-1} remains irreducible.

A conjectured answer to Problem I is the Mullineux map [6, 7] mentioned above which we describe in detail in Section 1. Mullineux defined a bijection between the set of p-regular partitions and a set of symbols

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix}, \qquad a_1 \geq \cdots \geq a_k > 0,\ r_1 \geq \cdots \geq r_k > 0$$

satisfying certain conditions. He then defined a "conjugation" map on the set of such symbols, and it is conjectured that the corresponding bijection of the p-regular partitions is exactly $\lambda \to \lambda^P$. Results which support this conjecture may be found in [1, 2, 5, 6, 7].

A conjecture concerning Problem II was given by Jantzen and Seitz (see [4]). Suppose that the partition λ of n has α_1 parts equal to $l_1, \ldots, \alpha_t$ parts equal to l_t, where $l_1 > l_2 > \cdots > l_t$, $0 < \alpha_i \leq p-1$. Then ρ_λ restricted to S_{n-1} is irreducible if and only if

$$l_i - l_{i+1} + \alpha_i + \alpha_{i+1} \equiv 0 \bmod p \qquad \text{for } 1 \leq i \leq t-1. \qquad (*)$$

The if part of this conjecture is proved [4, Theorem 4.2] and the only if part is known to be true if $t \leq 3$ [4, Cor. 4.5]. A p-regular partition satisfying $(*)$ will be called an *S-partition*.

It is obvious that the classes of all p-regular partitions and of all S-partitions are closed under "removal of the first column," i.e., under the operation of subtracting 1 from all the parts. In Section 1 we show how this operation affects the corresponding Mullineux symbols.

This is used in Section 2 to give an explicit description of the Mullineux symbols of S-partitions. As a corollary we get that the set of S-partitions is closed under the Mullineux map (Proposition 4.2). This shows that the conjectured answers to Problems I and II are compatible: it is clear that if ρ_λ remains irreducible when restricted to S_{n-1} then the same is true for ρ_{λ^P}, which is ρ_λ times the sign character. Thus a necessary and sufficient condition on λ for ρ_λ to remain irreducible should also be fulfilled for λ^P.

1. A General Result

We start by describing the Mullineux map $\lambda \mapsto \lambda^M$ on the set $\operatorname{Par}_p(n)$ of p-regular partitions of n.

Let $\lambda \in \operatorname{Par}_p(n)$. The *p-rim* of λ is a part of the rim of λ [3, p. 56], which is composed of *p-segments*. Each p-segment except possibly the last contains p points. The first p-segment consists of the first p points of the rim of λ, starting with the longest row. (If the rim contains at most p points it is the entire rim.) The next segment is obtained by starting in the row next below the previous p-segment. This process is continued until the final row is reached. We let a_1 be the number of nodes in the p-rim of $\lambda = \lambda^{(1)}$ and let r_1 be the number of rows in λ. Removing the p-rim of $\lambda = \lambda^{(1)}$ we get a new p-regular partition $\lambda^{(2)} \in \operatorname{Par}_p(n - a_1)$. We let a_2, r_2 be the length of the p-rim and the number of parts of $\lambda^{(2)}$, respectively. Continuing in this way we get a sequence of partitions $\lambda = \lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)}$ where $\lambda^{(k)} \neq 0$ and $\lambda^{(k+1)} = 0$, and a corresponding *Mullineux symbol* of λ

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix}.$$

The integer k is called the *length* of the symbol.

It is easy to recover λ from its Mullineux symbol $G_p(\lambda)$. (Start with the hook $\lambda^{(k)}$, given by a_k, r_k, and work backwards. In placing each p-rim it is convenient to start from below, at row r_i). Moreover, by a slight reformulation of a result in [6], the entries of $G_p(\lambda)$ satisfy (see [1])

$$\begin{cases} (1) & \varepsilon_i \leq r_i - r_{i+1} < p + \varepsilon_i,\ 1 \leq i \leq k-1;\ 1 \leq r_k < p \\ (2) & r_i - r_{i+1} + \varepsilon_{i+1} \leq a_i - a_{i+1} < p + r_i - r_{i+1} + \varepsilon_{i+1};\ 1 \leq i \leq k-1; \\ & r_k \leq a_k < p + r_k \\ (3) & \sum_i a_i = n, \end{cases} \tag{1.1}$$

where $\varepsilon_i = 1$ if $p \nmid a_i$, $\varepsilon_i = 0$ if $p \mid a_i$.

If $G_p(\lambda)$ is as above then $\lambda^M \in \operatorname{Par}_p(n)$ is by definition the partition satisfying

$$G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix}, \qquad \text{where } s_i = a_i - r_i + \varepsilon_i \tag{1.2}$$

Example. $p = 5$, $\lambda = (8, 6, 5^2)$

$$\begin{array}{cccccccc} 4 & 4 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 1 & 1 & & \\ 3 & 3 & 2 & 2 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & & \end{array} \qquad G_5(\lambda) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 4 & 4 & 3 & 2 \end{pmatrix}$$

$$\begin{array}{cccccccccc} 4 & 4 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & & \\ 2 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ 1 & & & & & & & & & \end{array} \qquad G_5(\lambda^M) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 6 & 3 & 2 & 2 \end{pmatrix}$$

(In both cases the nodes of the successive 5-rims are numbered 1, 2, 3, 4.) Thus $(8, 6, 5^2)^M = (10, 8, 2^2, 1^2)$.

PROPOSITION 1.3. *Suppose that the p-regular partition* $\lambda = (l_1, l_2, \dots, l_t)$ *has*

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix}$$

as its Mullineux symbol and that $\mu = (l_1 - 1, l_2 - 1, \dots, l_t - 1)$ (*obtained by removing the first column from* λ) *has*

$$G_p(\mu) = \begin{pmatrix} a'_1 & \cdots & a'_l \\ r'_1 & \cdots & r'_l \end{pmatrix}$$

as its Mullineux symbol of length l (*say*). *Then*

$$\begin{cases} l = k & \text{if } a_k \neq r_k \\ l = k - 1 & \text{if } a_k = r_k \\ \text{and for } i \geq 1 & \\ a'_i = a_i - r_i + r_{i+1} & \\ r'_i = r_{i+1} + \delta_i & \end{cases} \qquad (*)$$

where

$$\delta_i = \begin{cases} 0 & p \mid a'_i \\ 1 & p \nmid a'_i \end{cases}$$

and we have put $r_{k+1} = 0$.

Proof. Induction on k. $k = 1$. Here

$$\begin{pmatrix} a_1 \\ r_1 \end{pmatrix} \text{ is the Mullineux symbol of } \lambda = \left(a_1 - r_1 + 1, 1^{r_1 - 1}\right).$$

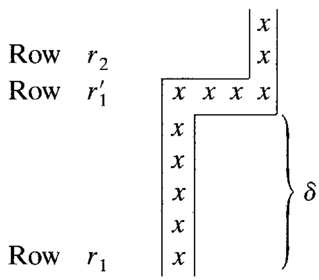
Then $\mu = (a_1 - r_1)$ has the Mullineux symbol

$$\begin{cases} \begin{pmatrix} a_1 - r_1 \\ 1 \end{pmatrix} & \text{if } a_1 \neq r_1 \text{ (note } a_1 - r_1 < p) \\ \varnothing & \text{if } a_1 = r_1, \end{cases}$$

so the proposition is true in this case.

We denote generally by $A_1(\lambda)$ and $A_1(\mu)$ the p-rims of λ and μ, respectively. Obviously $A_1(\mu) \subseteqq A_1(\lambda)$ and it is clear that the partition $\mu_1 = \mu \setminus A_1(\mu)$ is obtained from $\lambda_1 = \lambda \setminus A_1(\lambda)$ by removing the first column. By induction $(*)$ is therefore fulfilled for $i \geq 2$. We consider the case $i = 1$. Let δ be the multiplicity of 1 in λ. The node in the first column and the $(\delta + 1)$th last row of λ is called the *critical* node.

Case 1. $A_1(\lambda)$ contains the critical node



In this case $A_1(\lambda)$ contains all nodes in the $(\delta + 1)$th last row of λ. This means that

$$r_1 - r_2 = \delta + 1 = a_1 - a_1'$$

whereas

$$r_1 - r_1' = \delta.$$

We get easily

$$a_1' = a_1 - r_1 + r_2 \qquad \text{and} \qquad r_1' = r_2 + 1.$$

Note that in this case $p \nmid a_1'$. Indeed, if $p | a_1'$ then $A_1(\mu)$ would have the last segment of length p and this segment would have to end in the

$(\delta + 1)$th last row of λ. This obviously would contradict the fact that $A_1(\lambda)$ contains the critical node.

Case 2. $A_1(\lambda)$ does not contain the critical node.

$$\begin{array}{lccc}
 & & & x \\
 & & & x \\
\text{Row } r_1' = r_2 & & x & x \\
 & x & & \\
 & x & & \delta \\
\text{Row } \quad r_1 & x & &
\end{array}$$

In this case the second-to-last segment of $A_1(\lambda)$ ends in the $(\delta + 1)$th last row in λ and we get that $A_1(\mu)$ consists of all but the last segment of $A_1(\lambda)$. Thus $p|a_1'$. Moreover

$$r_1 - r_2 = r_1 - r_1' = a_1 - a_1' = \delta$$

in this case. We get

$$a_1' = a_1 - r_1 + r_2 \qquad \text{and} \qquad r_1 = r_1'$$

as desired.

COROLLARY 1.4. *In the notation of Proposition* 1.3 *we have for the multiplicity* δ *of* 1 *in* λ

$$\delta = r_1 - r_2 - \varepsilon,$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } a_1 \equiv r_1 - r_2 \pmod p \\ 1 & \text{otherwise}. \end{cases}$$

COROLLARY 1.5. *In the notation of Proposition* 1.3 *let* δ *and* δ^M *be the multiplicity of* 1 *in* λ *and* λ^M, *respectively. Then*

$$a_1 - a_2 - 1 \le \delta + \delta^M \le a_1 - a_2 + 1.$$

Proof. Apply Corollary 1.4 to λ and to λ^M and combine with (1.2).

PROPOSITION 1.6. *Assume that the p-regular partition* $\mu = (l_1, l_2, \ldots, l_t)$ *has the Mullineux symbol*

$$G_p(\mu) = \begin{pmatrix} b_1 & \cdots & b_k \\ s_1 & \cdots & s_k \end{pmatrix}.$$

Let $0 \le \delta \le p-1$. *Then* $\lambda = (l_1+1, l_2+1, \ldots, l_t+1, 1^\delta)$, *a partition obtained by adding a column to* μ, *has the Mullineux symbol*

$$G_p(\lambda) = \begin{pmatrix} b_1+\delta+\varepsilon_1 & b_2-s_2+\varepsilon_2+s_1-\varepsilon_1 & \cdots & b_k-s_k+\varepsilon_k+s_{k-1}-\varepsilon_{k-1} & s_k-\varepsilon_k \\ s_1+\delta & s_1-\varepsilon_1 & \cdots & s_{k-1}-\varepsilon_{k-1} & s_k-\varepsilon_k \end{pmatrix}$$

where

$$\varepsilon_i = \begin{cases} 0 & \text{if } p \mid b_i \\ 1 & \text{if } p \nmid b_i \end{cases}$$

(*and the column* $\begin{smallmatrix} s_k-\varepsilon_k \\ s_k-\varepsilon_k \end{smallmatrix}$ *is omitted if* $s_k = \varepsilon_k = 1$).

Proof. By Proposition 1.3 the Mullineux symbol of λ has the form

$$\begin{pmatrix} a_1 & \cdots & a_{k+1} \\ r_1 & \cdots & r_{k+1} \end{pmatrix},$$

where we assume that $a_{k+1} = r_{k+1}$ and for convenience do not exclude the possibility $a_{k+1} = 0$. Since μ has s_1 rows, λ has $s_1+\delta$ rows, whence

$$r_1 = s_1 + \delta.$$

For $i \ge 1$ Proposition 1.3 shows that $s_i = r_{i+1} + \varepsilon_i$, whence $r_2 = s_1 - \varepsilon_1, \ldots, r_{k+1} = s_k - \varepsilon_k$. Using the equations

$$b_i = a_i + r_{i+1} - r_i, \qquad \text{i.e., } a_i = b_i - r_{i+1} + r_i,$$

which hold by Proposition 1.3, we get our result.

2. Mullineux Symbols of S-Partitions

We are concerned with two sets $\mathcal{S}$ and $\mathcal{T}$ of Mullineux symbols, defined in different ways, and the main result is that these sets are equal. As before, a p-regular partition $\lambda = (l_1^{\alpha_1}, \ldots, l_t^{\alpha_t})$, $l_1 > l_2 > \cdots > l_t$, $0 < \alpha_i < p$ is called an *S-partition* if $l_i - l_{i+1} + \alpha_i + \alpha_{i+1} \equiv 0 \bmod p$ for $1 \le i \le t-1$.

The first set of symbols is then

$$\mathcal{S} = \{G_p(\lambda) \mid \lambda \text{ is an } S\text{-partition}\}.$$

We define the *type* $\alpha = \alpha(\lambda)$ of an S-partition to be the integer α, $0 \leq \alpha \leq p-1$, such that $l_1 - \alpha_1 \equiv \alpha \pmod p$.

For $0 \leq \alpha \leq p-1$ we let

$$\mathscr{S}_\alpha = \{G_p(\lambda) | \lambda \text{ is an } S\text{-partition of type } \alpha\}$$

so that

$$\mathscr{S} = \mathscr{S}_0 \cup \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{p-1} \qquad \text{(disjoint)}.$$

For later use we note:

LEMMA 2.1. *If the S-partition $\mu \neq \varnothing$ is obtained from the S-partition λ by removing its first column, then*

$$\alpha(\mu) \equiv \alpha(\lambda) - 1 \pmod p.$$

LEMMA 2.2. *Let $\lambda = (l_1^{\alpha_1}, \ldots, l_t^{\alpha_t})$ be an S-partition of type α as above and*

$$G_p(\lambda) = \begin{pmatrix} a_1 & \cdots & a_k \\ r_1 & \cdots & r_k \end{pmatrix}.$$

Then

$$l_t + \alpha_t \equiv \alpha + 2r_1 \pmod p.$$

Proof. By definition, $\alpha \equiv l_1 - \alpha_1 \pmod p$. Since λ is an S-partition

$$l_i - \alpha_i \equiv l_{i+1} - \alpha_{i+1} - 2\alpha_i \pmod p, \qquad \text{for } 1 \leq i \leq t-1.$$

Thus

$$\begin{aligned} \alpha &\equiv l_1 - \alpha_1 \equiv l_2 - \alpha_2 - 2\alpha_1 \equiv l_3 - \alpha_3 - 2(\alpha_1 + \alpha_2) \equiv \ldots \\ &\equiv l_t - \alpha_t - 2(\alpha_1 + \cdots + \alpha_{t-1}) \pmod p. \end{aligned}$$

As r_1 is the number of parts of λ we have

$$\alpha_1 + \cdots + \alpha_{t-1} = r_1 - \alpha_t.$$

Thus

$$\alpha \equiv l_t - \alpha_t - 2(r_1 - \alpha_t) = l_t + \alpha_t - 2r_1 \pmod p,$$

proving the lemma.

We proceed to describe the second set of symbols $\mathscr{T}$, which is to be the disjoint union of sets $\mathscr{T}_0, \mathscr{T}_1, \ldots, \mathscr{T}_{p-1}$. Let $0 \leq \alpha \leq p-1$. We describe the symbols in $\mathscr{T}_\alpha$ inductively according to their length. First we observe that it suffices to give the entries a_i, r_i in a symbol modulo p, since the conditions (1.1) for Mullineux symbols then determine them completely. We call a column $\begin{smallmatrix} a_i \\ r_i \end{smallmatrix}$ *singular* if $a_i \equiv 0 \pmod p$ and *regular* otherwise.

The symbols of length 1 in $\mathscr{T}_\alpha$ are

$$\begin{pmatrix} p \\ p-\alpha \end{pmatrix}, \qquad \text{if } \alpha \neq 0$$

$$\begin{pmatrix} p+1-\alpha \\ p+1-\alpha \end{pmatrix} \qquad \text{if } \alpha \neq 0, 1$$

$$\begin{pmatrix} \alpha+1 \\ 1 \end{pmatrix} \qquad \text{if } \alpha \neq p-1.$$

Suppose now that $\left(\begin{smallmatrix} a_2 & \cdots & a_k \\ r_2 & \cdots & r_k \end{smallmatrix}\right)$ is a Mullineux symbol in $\mathscr{T}_\alpha$ of length $k-1 \geq 2$. We describe its possible extensions to a Mullineux symbol $\left(\begin{smallmatrix} a_1 & \cdots & a_k \\ r_1 & \cdots & r_k \end{smallmatrix}\right)$ in $\mathscr{T}_\alpha$. We notice that the symbols of length 1 in $\mathscr{T}_\alpha$ consisting of a regular column both satisfy

$$a_1 - 2r_1 + 1 \equiv \alpha \pmod p, \tag{$*$}$$

whereas the symbol consisting of a singular column satisfies

$$a_1 - r_1 \equiv \alpha \pmod p.$$

We will define the extensions in such a way that all the regular columns satisfy $(*)$:

Regular extensions in $\mathscr{T}_\alpha$:

$$(\mathrm{R}_1) \qquad \begin{pmatrix} \alpha+2r_2+1 & a_2 & \cdots & a_k \\ r_2+1 & r_2 & \cdots & r_k \end{pmatrix} \qquad \text{if } \alpha+2r_2+1 \not\equiv 0$$

$$(\mathrm{R}_2) \qquad \begin{pmatrix} -\alpha-2r_2+1 & a_2 & \cdots & a_k \\ -\alpha-r_2+1 & r_2 & \cdots & r_k \end{pmatrix} \qquad \text{if } \alpha+2r_2-1 \not\equiv 0$$

Singular extensions in $\mathscr{T}_\alpha$:

$$(\mathrm{S}_1) \qquad \begin{pmatrix} 0 & a_2 & \cdots & a_k \\ r_2 & r_2 & \cdots & r_k \end{pmatrix}$$

$$(\mathrm{S}_2) \qquad \begin{pmatrix} 0 & a_2 & \cdots & a_k \\ -\alpha-r_2 & r_2 & \cdots & r_k \end{pmatrix}.$$

In fact, starting with the degenerate column $\begin{smallmatrix}0\\0\end{smallmatrix}$, these rules also give the symbols of length 1; more precisely

$$(\mathrm{R}_1) \qquad \text{gives } \begin{pmatrix}\alpha+1\\1\end{pmatrix} \qquad \text{for } \alpha+1 \not\equiv 0$$

$$(\mathrm{R}_2) \qquad \text{gives } \begin{pmatrix}1-\alpha\\1-\alpha\end{pmatrix} \qquad \text{for } \alpha-1 \not\equiv 0$$

$$(\mathrm{S}_2) \qquad \text{gives } \begin{pmatrix}0\\-\alpha\end{pmatrix} \qquad (\text{degenerates for } \alpha = 0)$$

(keep in mind that we are using the convention of writing the entries modulo p).

We refer to the symbols in $\mathscr{T}_\alpha$ as symbols in $\mathscr{T}$ of *type* α. For $n \in \mathbb{N}$ we set

$$\mathscr{S}_\alpha(n) = \{G_p(\lambda) \in \mathscr{S}_\alpha | \lambda \vdash n\}$$
$$\mathscr{T}_\alpha(n) = \{G_p(\lambda) \in \mathscr{T}_\alpha | \lambda \vdash n\}$$

so that $\mathscr{S}_\alpha = \bigcup_{n\in\mathbb{N}} \mathscr{S}_\alpha(n)$, $\mathscr{T}_\alpha = \bigcup_{n\in\mathbb{N}} \mathscr{T}_\alpha(n)$.

EXAMPLE. To illustrate the construction method used for defining $\mathscr{T}_\alpha$, we let $p = 5$ and start with the symbol $\left(\begin{smallmatrix}3\\1\end{smallmatrix}\right)$ of length 1 and type 2, i.e., an element in $\mathscr{T}_2$. What are its regular extensions? As only the condition for (R_2) is satisfied, we can only form one regular extension; according to the definition of (R_2) and the restrictions for Mullineux symbols, we obtain $\left(\begin{smallmatrix}7&3\\3&1\end{smallmatrix}\right)$. Furthermore, there are two singular extensions of $\left(\begin{smallmatrix}3\\1\end{smallmatrix}\right)$. The (S_1) construction rule gives $\left(\begin{smallmatrix}5&3\\1&1\end{smallmatrix}\right)$, and by the (S_2) rule we also get $\left(\begin{smallmatrix}5&3\\2&1\end{smallmatrix}\right)$.

The main result of this article is now the following.

THEOREM 2.3. *For all* $\alpha \in \{0, \dots, p-1\}$ *and* $n \in \mathbb{N}$, *we have*

$$\mathscr{S}_\alpha(n) = \mathscr{T}_\alpha(n).$$

The proof of this theorem will occupy the next section.

3. PROOF OF THE THEOREM

First we have to prove some technical lemmas which will be used in the induction step of the proof of the theorem.

LEMMA 3.1. *If* $G_p(\mu) \in \mathscr{S}_{\alpha-1} \cap \mathscr{T}_{\alpha-1}$ *and* λ *is obtained from* μ *by adding a column (as in Proposition* (1.6)) *such that* $G_p(\lambda) \in \mathscr{T}_\alpha$, *then* $G_p(\lambda) \in \mathscr{S}_\alpha$.

Proof. Using the notation of Proposition 1.6, $G_p(\lambda) \in \mathscr{S}_\alpha$ is obvious if $\delta = 0$. If $\delta \neq 0$, we only have to prove the Seitz condition at "the end" of λ, i.e., we have to show

$$l_t + \alpha_t + \delta \equiv 0 \pmod{p}.$$

Now if $b_1 + \varepsilon_1 + \delta \equiv 0$, then we must have $b_1 \not\equiv 0$ and as $G_p(\mu) \in \mathscr{T}_{\alpha-1}$, this implies $b_1 - 2s_1 + 1 \equiv \alpha - 1 \pmod{p}$. Using Lemma 2.2 we now obtain

$$l_t + \alpha_t + \delta \equiv \alpha - 1 + 2s_1 + \delta \equiv b_1 + 1 + \delta \equiv 0.$$

If $b_1 + \varepsilon_1 + \delta \not\equiv 0$, then $G_p(\lambda) \in \mathscr{T}_\alpha$ implies

$$b_1 + \varepsilon_1 + \delta - 2(s_1 + \delta) + 1 \equiv b_1 + \varepsilon_1 - 2s_1 + 1 - \delta \equiv \alpha,$$

so $l_t + \alpha_t + \delta \equiv b_1 + \varepsilon_1$ (again by Lemma 2.2). Now if $b_1 \not\equiv 0$, then we obtain the contradiction $\delta \equiv 0$, hence $b_1 \equiv 0$ and thus $l_t + \alpha_t + \delta \equiv 0$.

LEMMA 3.2. *If* $G_p(\lambda) \in \mathscr{T}_\alpha$ *and* $\mu \neq \varnothing$ *is obtained from* λ *by removing the first column from* λ, *then* $G_p(\mu) \in \mathscr{T}_{\alpha-1}$.

Proof. Let $G_p(\lambda) = \begin{pmatrix} a_1 & \cdots & a_k \\ r_1 & \cdots & r_k \end{pmatrix}$; by Proposition 1.3 we know that

$$G_p(\mu) = \begin{pmatrix} a'_1 & \cdots & a'_k \\ r'_1 & \cdots & r'_k \end{pmatrix} \quad \text{with} \quad \begin{matrix} a'_i = a_i - r_i + r_{i+1} \\ r'_i = r_{i+1} + \delta_i \end{matrix}, 1 \le i \le k,$$

where

$$r_{k+1} = 0, \qquad \delta_i = \begin{cases} 0 & \text{if } p \mid a'_i \\ 1 & \text{if } p \nmid a'_i \end{cases}.$$

For convenience we allow a redundant column $\begin{matrix} a'_k \\ r'_k \end{matrix} = \begin{matrix} 0 \\ 0 \end{matrix}$ at the end. We prove the lemma by induction on k.

$k = 1$. We have $G_p(\lambda) = \begin{pmatrix} p \\ p-\alpha \end{pmatrix}$, $\alpha \neq 0$ or $G_p(\lambda) = \begin{pmatrix} \alpha+1 \\ 1 \end{pmatrix}$, $\alpha \neq 0$, $p - 1$. In both cases $G_p(\mu) = \begin{pmatrix} \alpha \\ 1 \end{pmatrix}$, which is in $\mathscr{T}_{\alpha-1}$.

$k = 2$. From now on we use systematically the observation that it suffices to know the entries in the Mullineux symbol modulo p and work only with congruence classes. To avoid confusion we denote the mod p version of the Mullineux symbol $G_p(\lambda)$ by $G_{(p)}(\lambda)$. We now consider all the possible cases. Of course the case labels refer to the previous section; as there is only one singular symbol of length 1, we denote a singular rightmost column in $G_p(\lambda)$ just by S.

(i) R_1S *and* S_2S. For

$$G_{(p)}(\lambda) = \begin{pmatrix} 1-\alpha & 0 \\ 1-\alpha & -\alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{with } \alpha \neq 0$$

we obtain $G_{(p)}(\mu) = \begin{pmatrix} -\alpha & \alpha \\ 1-\alpha & 1 \end{pmatrix}$, which is in $\mathscr{T}_{\alpha-1}$ since $-\alpha \equiv -(\alpha-1)-2+1$ and $-\alpha+1 \equiv -(\alpha-1)-1+1$. Thus $G_p(\mu)$ is of the form R_2R_1.

(ii) R_2S *and* S_1S. For

$$G_{(p)}(\lambda) = \begin{pmatrix} \alpha+1 & 0 \\ 1 & -\alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ -\alpha & -\alpha \end{pmatrix} \quad \text{with } \alpha \neq 0$$

we obtain $G_{(p)}(\mu) = \begin{pmatrix} 0 & \alpha \\ -\alpha & 1 \end{pmatrix}$, which is in $\mathscr{T}_{\alpha-1}$ since $-\alpha \equiv -(\alpha-1)-1$. Thus $G_p(\mu)$ is of the form S_2R_1.

(iii) R_1R_2 *and* S_2R_2. For

$$G_{(p)}(\lambda) = \begin{pmatrix} 3-\alpha & 1-\alpha \\ 2-\alpha & 1-\alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1-\alpha \\ -1 & 1-\alpha \end{pmatrix} \quad \text{with } \alpha \neq 0, 1$$

we obtain

$$G_{(p)}(\mu) = \begin{pmatrix} 2-\alpha \\ 2-\alpha \end{pmatrix} \text{ for } \alpha \neq 2 \qquad \text{and} \qquad G_{(p)}(\mu) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \text{ for } \alpha = 2.$$

These are in $\mathscr{T}_{\alpha-1}$ of the form R_2 resp. S.

(iv) R_2R_2 *and* S_1R_2. For

$$G_{(p)}(\lambda) = \begin{pmatrix} \alpha-1 & 1-\alpha \\ 0 & 1-\alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1-\alpha \\ 1-\alpha & 1-\alpha \end{pmatrix} \quad \text{with } \alpha \neq 0, 1$$

we obtain $G_{(p)}(\mu) = \begin{pmatrix} 0 \\ 1-\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ -(\alpha-1) \end{pmatrix} \in \mathscr{T}_{\alpha-1}$.

(v) R_1R_1 *and* S_2R_1. For

$$G_{(p)}(\lambda) = \begin{pmatrix} \alpha+3 & \alpha+1 \\ 2 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \alpha+1 \\ -\alpha-1 & 1 \end{pmatrix} \quad \text{with } \alpha \neq p-1$$

we obtain

$$G_{(P)}(\mu) = \begin{cases} \begin{pmatrix} \alpha+2 & \alpha \\ 2 & 1 \end{pmatrix} & \text{for } \alpha \neq 0,\, p-2 \\ \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} & \text{for } \alpha = p-2 \\ \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \text{for } \alpha = 0. \end{cases}$$

All of these are in $\mathcal{T}_{\alpha-1}$ of the form R_1R_1, S_1R_1, and R_2, respectively.

(vi) R_2R_1 *and* S_1R_1.

$$G_{(p)}(\lambda) = \begin{pmatrix} -\alpha-1 & \alpha+1 \\ -\alpha & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \alpha+1 \\ 1 & 1 \end{pmatrix},\ \alpha \neq p-1,$$

both giving

$$G_{(p)}(\mu) = \begin{pmatrix} 0 & \alpha \\ 1 & 1 \end{pmatrix},\ \alpha \neq 0 \quad \text{and} \quad G_{(p)}(\mu) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \alpha = 0.$$

These are in $\mathcal{T}_{\alpha-1}$ of the form S_1R_1, resp. S.

$k \geq 3$. Removing the first p-rim gives a partition $\tilde{\lambda}$ with

$$G_p(\tilde{\lambda}) = \begin{pmatrix} a_2 & \cdots & a_k \\ r_2 & \cdots & r_k \end{pmatrix} \in \mathcal{T}_\alpha.$$

By the induction hypotheses, removing the first column of $\tilde{\lambda}$ results in a partition $\tilde{\mu}$ with

$$G_p(\tilde{\mu}) = \begin{pmatrix} a'_2 & \cdots & a'_k \\ r'_2 & \cdots & r'_k \end{pmatrix} \in \mathcal{T}_{\alpha-1}$$

(as $k \geq 3$, $\tilde{\mu} \neq \varnothing$!).

Hence we only have to check that $\substack{a'_1 \\ r'_1}$ is added to $G_p(\tilde{\mu})$ according to the $\mathcal{T}_{\alpha-1}$ construction rules. To consider all possible cases, we have to look at the first three columns in $G_{(p)}(\lambda)$.

(i) $S_1S_1\ldots, R_2S_1\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 & a & \cdots \\ r & r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 1-\alpha-2r & 0 & a & \cdots \\ 1-\alpha-r & r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{pmatrix} 0 & 0 & \cdots \\ r & r & \cdots \end{pmatrix} \in \mathscr{T}_{\alpha-1},$$

as the first column is added according to the (S_1)-rule.

(ii) $S_1R_2\ldots, R_2R_2\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 1-\alpha-2r & a & \cdots \\ 1-\alpha-r & 1-\alpha-r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} -1+\alpha+2r & 1-\alpha-2r & a & \cdots \\ r & 1-\alpha-r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{pmatrix} 0 & 0 & \cdots \\ 1-\alpha-r & r & \cdots \end{pmatrix} \in \mathscr{T}_{\alpha-1},$$

as this is an extension according to the (S_2)-rule

(iii) $S_2S_1\ldots, R_1S_1\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 & a & \cdots \\ -\alpha-r & r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 1+\alpha+2r & 0 & a & \cdots \\ 1+r & r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} \alpha+2r & 0 & \cdots \\ r+1 & r & \cdots \end{pmatrix} & \text{if } \alpha+2r \not\equiv 0 \\ \begin{pmatrix} 0 & 0 & \cdots \\ r & r & \cdots \end{pmatrix} & \text{if } \alpha+2r \equiv 0. \end{cases}$$

This is in $\mathscr{T}_{\alpha-1}$ (R_1-resp. S_1-rule).

(iv) $S_2R_2\ldots, R_1R_2\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 1-\alpha-2r & a & \cdots \\ r-1 & 1-\alpha-r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 3-\alpha-2r & 1-\alpha-2r & a & \cdots \\ 2-\alpha-r & 1-\alpha-r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} -\alpha - 2r & 0 & \cdots \\ 2 - \alpha - r & r & \cdots \end{pmatrix} & \text{if } \alpha + 2r \not\equiv 0 \\ \begin{pmatrix} 0 & 0 & \cdots \\ 1 - \alpha - r & r & \cdots \end{pmatrix} & \text{if } \alpha + 2r \equiv 0. \end{cases}$$

This is in $\mathcal{T}_{\alpha-1}$ (R_2-resp. S_2-rule).

(v) $S_2S_2\ldots, R_1S_2\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 & a & \cdots \\ r & -\alpha - r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 1 - \alpha - 2r & 0 & a & \cdots \\ 1 - \alpha - r & -\alpha - r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} -\alpha - 2r & \alpha + 2r & \cdots \\ 1 - \alpha - r & r + 1 & \cdots \end{pmatrix} & \text{if } \alpha + 2r \not\equiv 0 \\ \begin{pmatrix} 0 & 0 & \cdots \\ -\alpha - r & r & \cdots \end{pmatrix} & \text{if } \alpha + 2r \equiv 0. \end{cases}$$

This is in $\mathcal{T}_{\alpha-1}$ (R_2-resp. S_1-rule).

(vi) $S_1S_2\ldots, R_2S_2\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 & \cdots & \\ -\alpha - r & -\alpha - r & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 1 + \alpha + 2r & 0 & \cdots & \\ 1 + r & -\alpha - r & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} 0 & \alpha + 2r & \cdots \\ -\alpha - r & r + 1 & \cdots \end{pmatrix} & \text{if } \alpha + 2r \not\equiv 0 \\ \begin{pmatrix} 0 & 0 & \cdots \\ -\alpha - r & r & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots \\ r & r & \cdots \end{pmatrix} & \text{if } \alpha + 2r \equiv 0 \end{cases}$$

This is in $\mathcal{T}_{\alpha-1}$ (S_2-resp. S_1-rule).

(vii) $S_1R_1\ldots, R_2R_1\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & \alpha+2r+1 & \cdots & \\ r+1 & r+1 & r & \cdots \end{pmatrix} \\ \begin{pmatrix} -1-\alpha-2r & \alpha+2r+1 & \cdots & \\ -\alpha-r & r+1 & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} 0 & \alpha+2r & \cdots \\ r+1 & r+1 & \cdots \end{pmatrix} & \alpha+2r \not\equiv 0 \\ \begin{pmatrix} 0 & 0 & \cdots \\ r+1 & r & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots \\ 1-\alpha-r & r & \cdots \end{pmatrix} & \alpha+2r \equiv 0 \end{cases}$$

This is in $\mathscr{T}_{\alpha-1}$ (S_1-resp. S_2-rule).

(viii) $S_2R_1\ldots, R_1R_1\ldots.$

$$G_{(p)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & \alpha+2r+1 & \cdots & \\ -1-\alpha-r & r+1 & r & \cdots \end{pmatrix} \\ \begin{pmatrix} 3+\alpha+2r & \alpha+2r+1 & \cdots & \\ 2+r & r+1 & r & \cdots \end{pmatrix} \end{cases}$$

implies

$$G_{(p)}(\mu) = \begin{cases} \begin{pmatrix} 2+\alpha+2r & \alpha+2r & \cdots \\ r+2 & r+1 & \cdots \end{pmatrix} & \alpha+2r \not\equiv 0, -2 \\ \begin{pmatrix} 0 & -2 & \cdots \\ r+1 & r+1 & \cdots \end{pmatrix} & \alpha+2r \equiv -2 \\ \begin{pmatrix} 2 & 0 & \cdots \\ r+2 & r & \cdots \end{pmatrix} = \begin{pmatrix} 2-\alpha-2r & 0 & \cdots \\ 2-\alpha-r & 0 & \cdots \end{pmatrix} & \alpha+2r \equiv 0. \end{cases}$$

This is in $\mathscr{T}_{\alpha-1}$ (R_1-, S_1-, resp. R_2-rule).
We now prove the main result of this paper.

Proof of Theorem 2.3. The proof is by induction on n, the result being obvious for $n = 1$. Also, the symbols of length 1 are the same on both sides.

Let $G_p(\lambda) \in \mathscr{T}_\alpha(n)$ of length at least 2 and let $\mu \vdash m$ be the partition obtained from λ by removing the first column. Note that $\mu \neq \varnothing$. By Lemma 3.2 we have $G_p(\mu) \in \mathscr{T}_{\alpha-1}(m)$ and hence by induction $G_p(\mu) \in \mathscr{S}_{\alpha-1}(m)$. By Lemma 3.1 we then conclude that $G_p(\lambda) \in \mathscr{S}_\alpha(n)$. Hence we have proved $\mathscr{T}_\alpha(n) \subseteq \mathscr{S}_\alpha(n)$ for all n.

To prove the other inclusion we take $G_p(\lambda) \in \mathscr{S}_\alpha(n)$ of length ≥ 2. Again we remove its first column to obtain a partition $\mu \vdash m$, $\mu \neq \varnothing$ with $G_p(\mu) \in \mathscr{S}_{\alpha-1}(m)$. By induction $G_p(\mu) \in \mathscr{T}_{\alpha-1}(m)$. We now prove the following lemma to conclude $G_p(\lambda) \in \mathscr{T}_\alpha(n)$ and thus finish the proof.

LEMMA 3.3. *IF $G_p(\mu) \in \mathscr{T}_{\alpha-1}$ and λ is obtained from μ by adding a column such that $G_p(\lambda) \in \mathscr{S}_\alpha$, then $G_p(\lambda) \in \mathscr{T}_\alpha$.*

Proof. By induction on the length k of $G_p(\mu)$. Assume

$$G_p(\mu) = \begin{pmatrix} b_1 & \cdots & b_k \\ s_1 & \cdots & s_k \end{pmatrix},$$

so by Proposition 1.6

$$G_p(\lambda) = \begin{pmatrix} b_1+\varepsilon_1+\delta & b_2-s_2+\varepsilon_2+s_1-\varepsilon_1 & \cdots & b_k-s_k+\varepsilon_k+s_{k-1}+\varepsilon_{k-1} & s_k-\varepsilon_k \\ s_1+\delta & s_1-\varepsilon_1 & \cdots & s_{k-1}-\varepsilon_{k-1} & s_k-\varepsilon_k \end{pmatrix}$$

where

$$\varepsilon_i = \begin{cases} 0 & \text{if } p \mid b_i \\ 1 & \text{if } p \nmid b_i \end{cases}$$

and the last column is omitted if $s_k = \varepsilon_k = 1$. Furthermore, $\lambda = (1^\delta 2^{\beta_1} \ldots)$ with $0 \leq \delta \leq p-1$. In fact, as $G_p(\lambda) \in \mathscr{S}_\alpha$, we know from Lemma 2.2 that $\delta = 0$ or $\delta \equiv \alpha + 2(s_1+\delta) - 1$, i.e., $\delta \equiv 1 - \alpha - 2s_1$. Assume $\delta = 0$. If $b_1 \equiv 0$, then

$$G_{(p)}(\lambda) = \begin{pmatrix} 0 & \cdots & \\ s_1 & s_1 & \cdots \end{pmatrix}$$

and the first column is added according to the S_1-rule. If $b_1 \not\equiv 0$, then

$$G_{(p)}(\lambda) = \begin{pmatrix} b_1+1 & \cdots & \cdots \\ s_1 & s_1-1 & \cdots \end{pmatrix}$$

and $b_1 - 2s_1 + 1 \equiv \alpha - 1 \pmod p$ by our observation on the regular columns in $\mathscr{T}_{\alpha-1}$ symbols.

Now if $b_1 + 1 \equiv 0$, then the first column is added by the S_2-rule, as $-\alpha - (s_1 - 1) \equiv s_1 - b_1 - 1 \equiv s_1$. If $b_1 + 1 \not\equiv 0$, then it is added by the R_1-rule as

$$\alpha + 2(s_1 - 1) + 1 = \alpha + 2s_1 - 1 \equiv b_1 + 1.$$

Since the rest of the $G_p(\lambda)$-symbol does not depend on δ, we may replace δ by $1-\alpha-2s_1$, and if $1-\alpha-2s_1 \equiv 0$ we only have to check the second and third columns in $G_p(\lambda)$.

$k=1$. Then we have:

$G_{(p)}(\mu)$	$\begin{pmatrix} 0 \\ 1-\alpha \end{pmatrix}, \alpha \neq 1$	$\begin{pmatrix} 2-\alpha \\ 2-\alpha \end{pmatrix}, \alpha \neq 1,2$	$\begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \alpha \neq 0$
$\delta \equiv$	$\alpha - 1$	$\alpha - 3$	$-\alpha - 1$
$G_{(p)}(\lambda)$	$\begin{pmatrix} \alpha-1 & 1-\alpha \\ 0 & 1-\alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 1-\alpha \\ -1 & 1-\alpha \end{pmatrix}$	$\begin{pmatrix} 0 \\ -\alpha \end{pmatrix}$
Construction rule	R_2R_2	S_2R_2	S

Furthermore, all $G_p(\lambda)$ are of type α, hence are in $\mathscr{T}_\alpha$.

$k \geq 2$. Let $\tilde{\mu}$ be the partition with $G_p(\tilde{\mu}) = \begin{pmatrix} b_2 & \cdots & b_k \\ s_2 & \cdots & s_k \end{pmatrix}$, so $G_p(\tilde{\mu}) \in \mathscr{T}_{\alpha-1}$, and hence (by induction or by the first part of the proof of the theorem) $G_p(\tilde{\mu}) \in \mathscr{S}_{\alpha-1}$. Now we add a column to $\tilde{\mu}$ such that the resulting partition $\tilde{\lambda}$ satisfies

$$G_p(\tilde{\lambda}) = \begin{pmatrix} b_2+\varepsilon_2+\tilde{\delta} & b_3-s_3+\varepsilon_3+s_2-\varepsilon_2 & \cdots & s_k-\varepsilon_k \\ s_2+\tilde{\delta} & s_2-\varepsilon_2 & \cdots & s_k-\varepsilon_k \end{pmatrix} \in \mathscr{S}_\alpha,$$

i.e., we take $\tilde{\delta}=0$ or $\tilde{\delta} \equiv 1-\alpha-2s_2$. By induction on k, $G_p(\tilde{\lambda}) \in \mathscr{T}_\alpha$. Hence we only have to look at the first three columns in $G_p(\lambda)$ to check that $G_p(\lambda) \in \mathscr{T}_\alpha$.

Instead of going through all 32 cases of the form

$$R_iR_jR,\ S_iR_jR,\ R_iS_jR,\ R_iR_jS,\ R_iS_jS,\ S_iR_jS,$$
$$S_iS_jR,\ S_iS_jS \qquad (i,j \in \{1,2\})$$

for the first three columns of $G_p(\mu)$ (where R and S just stand for a regular or singular column, respectively), we will only discuss three typical cases and otherwise omit these straightforward but tedious calculations.

(i) R_1R_1R.

$$G_{(p)}(\mu) = \begin{pmatrix} \alpha+2s+2 & \alpha+2s & b & \cdots \\ s+2 & s+1 & s & \cdots \end{pmatrix}$$

so (remember $\delta \equiv 1 - \alpha - 2s_1 \equiv -3 - \alpha - 2s$)

$$G_{(p)}(\lambda) = \begin{pmatrix} 0 & \alpha + 2s + 1 & b + 1 & \cdots \\ -1 - \alpha - s & s + 1 & s & \cdots \end{pmatrix}$$

follows the S_2R_1-rule if $\alpha + 2s + 1 \not\equiv 0$. If $\alpha + 2s + 1 \equiv 0$, then $-\alpha - s \equiv s + 1$, so then $G_{(p)}(\lambda)$ is of S_2S_2-type.

(ii) R_2S_2R.

$$G_{(p)}(\mu) = \begin{pmatrix} \alpha + 2s & 0 & b & \cdots \\ 1 + s & 1 - \alpha - s & s & \cdots \end{pmatrix}$$

so ($\delta \equiv 1 - \alpha - 2s_1 \equiv 1 - \alpha - 2(1 + s) = -1 - \alpha - 2s$)

$$G_{(p)}(\lambda) = \begin{pmatrix} 0 & 2s + \alpha - 1 & 0 & \cdots \\ -\alpha - s & s & 1 - \alpha - s & \cdots \end{pmatrix}.$$

If $\alpha + 2s - 1 \not\equiv 0$, then this is constructed according to the S_2R_2-rule. If $\alpha + 2s - 1 \equiv 0$, then $s \equiv 1 - \alpha - s$ and $G_p(\lambda)$ follows the S_2S_1-rule.

(iii) R_1R_2S.

$$G_{(p)}(\mu) = \begin{pmatrix} 4 - \alpha - 2s & 2 - \alpha - 2s & 0 & \cdots \\ 3 - \alpha - s & 2 - \alpha - s & s & \cdots \end{pmatrix}$$

so ($\delta \equiv 1 - \alpha - 2s_1 \equiv 1 - \alpha - 2(3 - \alpha - s) = \alpha + 2s - 5$)

$$G_{(p)}(\lambda) = \begin{pmatrix} 0 & 3 - \alpha - 2s & 1 - \alpha - 2s & \cdots \\ -2 + s & 2 - \alpha - s & 1 - \alpha - s & \cdots \end{pmatrix}.$$

Now this is constructed according to

S_2R_1 if $\alpha \not\equiv 3 - 2s$

S_2S_2 if $\alpha \equiv 3 - 2s$ (as $2 - \alpha - s \equiv -\alpha - (1 - \alpha - s)$).

4. Consequences

As mentioned at the beginning, our original aim was to prove the compatibility of the Mullineux and Seitz conjectures; we can now deduce this from our theorem via the description of S-partitions by Mullineux symbols in $\mathcal{T}$.

PROPOSITION 4.1. *Let $0 \leq \alpha \leq p - 1$. Let λ be a p-regular partition such that $G_p(\lambda) \in \mathcal{T}_\alpha$. Then $G_p(\lambda^M) \in \mathcal{T}_{p-\alpha}$. (We have put $\mathcal{T}_p = \mathcal{T}_0$.)*

Proof. The symbols in the sets $\mathscr{T}_\alpha$ were described inductively according to their length k in Section 2. Therefore it suffices to prove the proposition for $k = 1$ and 2. If $k = 1$ the result is immediate from the definitions. For $k = 2$ we have to consider the same cases $(\mathrm{R}_1\mathrm{S}, \ldots, \mathrm{S}_1\mathrm{R}_1)$ as in the proof of Lemma 3.2, $k = 2$. In each of these cases, the symbol $G_{(p)}(\lambda)$ is given in Lemma 3.2. Using (1.2) it is easily checked in each case that $G_{(p)}(\lambda^M)$ is in $\mathscr{T}_{p-\alpha}$, corresponding to another case. For instance, if

$$G_{(p)}(\lambda) = \begin{pmatrix} \alpha+3 & \alpha+1 \\ 2 & 1 \end{pmatrix} \qquad (\text{case } \mathrm{R}_1\mathrm{R}_1 \text{ for } \mathscr{T}_\alpha)$$

then

$$G_{(p)}(\lambda^M) = \begin{pmatrix} \alpha+3 & \alpha+1 \\ \alpha+2 & \alpha+1 \end{pmatrix} \qquad (\text{case } \mathrm{R}_1\mathrm{R}_2 \text{ for } \mathscr{T}_{p-\alpha}).$$

PROPOSITION 4.2. *Let $0 \le \alpha \le p - 1$. Let λ be an S-partition of type α. Then λ^M is an S-partition of type $p - \alpha$ (type p means type 0). In particular, the set of S-partitions is closed under the Mullineux map.*

Proof. Let λ be an S-partition of type α. Then by definition $G_p(\lambda) \in \mathscr{S}_\alpha$. By Theorem 2.3 $G_p(\lambda) \in \mathscr{T}_\alpha$ and thus $G_p(\lambda^M) \in \mathscr{T}_{p-\alpha}$ by Proposition 4.1. Then $G_p(\lambda^M) \in \mathscr{S}_{p-\alpha}$ by Theorem 2.3.

We finally list a couple of congruences relating λ and λ^M, when λ is an S-partition.

COROLLARY 4.3. *Let $\lambda = (l_1^{\alpha_1}, \ldots, l_t^{\alpha_t})$ be an S-partition and let $\lambda^M = (n_1^{\beta_1}, \ldots, n_s^{\beta_2})$. We have*

(1) $l_1 + n_1 \equiv \alpha_1 + \beta_1 \pmod p$

(2) $l_t + \alpha_t + n_s + \beta_s \equiv 2(a_1 + \varepsilon_1) \pmod p$

where a_1 is the length of the p-rim of λ and $\varepsilon_1 = 1$ if $p \nmid a_1$, $\varepsilon_1 = 0$ if $p | a_1$.

Proof. (1) If λ is of type α, then by definition $l_1 - \alpha_1 \equiv \alpha \pmod p$. By Proposition 4.2 we get $n_1 - \beta_1 \equiv -\alpha \pmod p$ and (1) follows.

(2) Proved similarly, using Lemma 2.2.

Remarks. (1) Examples show that Corollary 4.3 (1) and (2) are not true for arbitrary p-regular partitions. The proof of Lemma 2.2 gives further congruences relating λ and λ^M, when they are S-partitions. For instance, if $t \geq 2$ and $s \geq 2$ then

$$l_2 + n_2 \equiv \alpha_2 + \beta_2 + 2(\alpha_1 + \beta_1)$$
$$\equiv \alpha_2 + \beta_2 + 2(l_1 + n_1) \pmod p.$$

(2) It should be noted that due to our Theorem and the inductive description of the $\mathscr{T}_\alpha$-sets we have that if μ is obtained by removing the p-rim from the S-partition λ, then μ is also an S-partition (of the same type as λ).

(3) Also using the description of the $\mathscr{T}_\alpha$-sets we have that if λ is an S-partition of type α and $\substack{a_i \\ r_i}$ is a regular column in $G_p(\lambda)$, then

$$a_i - 2r_i + 1 \equiv \alpha \pmod p.$$

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